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Photon-added coherent states as nonlinear coherent states

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Abstract. The states $|\alpha, m\rangle$, defined as $\hat{a}^{\dagger m}|\alpha\rangle$ up to a normalization constant and where m is a non-negative integer, are shown to be the eigenstates of $f(\hat{n}, m)\hat{a}$ where $f(\hat{n}, m)$ is a nonlinear function of the number operator $\hat{n}$. The explicit form of $f(\hat{n}, m)$ is constructed. The eigenstates of this operator for negative values of m are introduced. The properties of these states are discussed and compared with those of the state $|\alpha, m\rangle$. The eigenstates corresponding to the positive and negative values of m are shown to be the result of nonunitarily deforming the number states $|m\rangle$ and $|0\rangle$ respectively.

1. Introduction

Coherent states are important in many fields of physics [1, 2]. Coherent states $|\alpha\rangle$, defined as the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [3], have properties like the classical radiation field. There exist states of the electromagnetic field whose properties, like squeezing, higher-order squeezing, antibunching and sub-Poissonian statistics [4, 5], are strictly quantum mechanical in nature. These states are called nonclassical states. The coherent states define the limit between the classical and nonclassical behaviour of the radiation field as far as the nonclassical effects are considered. A generalization of the coherent states was done by q-deforming the basic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ [6, 7]. A further generalization is to define states that are eigenstates of the operator $f(\hat{n})\hat{a}$,

$$f(\hat{n})\hat{a}|f, \alpha\rangle = \alpha|f, \alpha\rangle \tag{1}$$

where $f(\hat{n})$ is a operator valued function of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. These eigenstates are called as nonlinear coherent states and they are nonclassical. In the linear limit, $f(\hat{n}) = 1$, the nonlinear coherent states become the usual coherent states $|\alpha\rangle$. The nonlinear coherent states were introduced, as f-coherent states, in connection with the study of the oscillator whose frequency depends on its energy [8]. A class of nonlinear coherent states can be realized physically as the stationary states of the centre-of-mass motion of a trapped ion [9]. These nonlinear coherent states exhibit nonclassical features like squeezing and self-splitting.

The photon-added coherent states $|\alpha, m\rangle$ [10] are defined as

$$|\alpha, m\rangle = \frac{\hat{a}^{\dagger m}|\alpha\rangle}{\sqrt{\langle\alpha|\hat{a}^m\hat{a}^{\dagger m}|\alpha\rangle}} \tag{2}$$

where m is a non-negative integer. The states $|\alpha, m\rangle$ exhibit nonclassical features like phase squeezing and sub-Poissonian statistics. These states are produced in the interaction of a two-level atom, with ground state $|g\rangle$ and excited state $|e\rangle$, with a cavity field initially prepared in the coherent state $|\alpha\rangle$ [10]. Let the initial state of the atom-field system is $|\alpha\rangle|e\rangle$ and the interaction Hamiltonian $\hat{H}$ be $\hbar g(\sigma^+\hat{a} + \sigma^-\hat{a}^\dagger)$. Here σ^+ is the flip-operator corresponding to the atomic transition $|g\rangle \rightarrow |e\rangle$ and its conjugate σ^- corresponds to the transition from the excited state to the ground state. If the coupling constant g is small, the state of the atom field at time t, such that $gt << 1$, can be written as

$$|\psi(t)\rangle \simeq |\alpha\rangle|e\rangle - \frac{\mathrm{i}\hat{H}t}{\hbar}|\alpha\rangle|e\rangle. \tag{3}$$

Using the interaction Hamiltonian $\hat{H} = \hbar g(\sigma^+\hat{a} + \sigma^-\hat{a}^\dagger)$, the state $\psi(t)$ becomes $|\alpha\rangle|e\rangle - \mathrm{i}gt\hat{a}^\dagger|\alpha\rangle|g\rangle$. If the atom is detected to be in the ground state $|g\rangle$, then the field is in the state $\hat{a}^\dagger|\alpha\rangle$ which is the one-photon-added coherent state $|\alpha, 1\rangle$. If the interaction is a multiphoton process, $\hat{a}$ $(\hat{a}^\dagger) \rightarrow \hat{a}^m$ $(\hat{a}^{\dagger m})$, the m-photon-added coherent state $|\alpha, m\rangle$ can be produced.

In this paper it is shown that the photon-added coherent states can be interpreted as nonlinear coherent states. This is done by showing that the states $|\alpha, m\rangle$ obey the equation

$$f(\hat{n}, m)\hat{a}|\alpha, m\rangle = \alpha|\alpha, m\rangle \tag{4}$$

with a suitable choice for the function $f(\hat{n}, m)$. The operator $f(\hat{n}, m)$ is also a valid operator for negative values of m also. The eigenstates of $f(\hat{n}, m)\hat{a}$ with negative values of m are constructed and their properties compared with those of $|\alpha, m\rangle$.

2. Construction of $f(\hat{n}, m)$

In this section we construct the explicit form of the operator valued function $f(\hat{n}, m)$. The coherent states $|\alpha\rangle$ satisfy, by definition,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{5}$$

where α is a complex number. Premultiplying both the sides of this equation by $\hat{a}^{\dagger m}$ yields

$$\hat{a}^{\dagger m}\hat{a}|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle \tag{6}$$

where m is a non-negative integer. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, the above equation is written as

$$(\hat{a}\hat{a}^{\dagger m} - m\hat{a}^{\dagger(m-1)})|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle \tag{7}$$

which, making use of the identity $\frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a}\hat{a}^\dagger = 1$, leads to

$$\left(\hat{a} - \frac{m}{1+\hat{a}^\dagger\hat{a}}\hat{a}\right)\hat{a}^{\dagger m}|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle. \tag{8}$$

Comparing equation (4) with (8) gives the expression for $f(\hat{n}, m)$ as

$$f(\hat{n}, m) = 1 - \frac{m}{1+\hat{a}^\dagger\hat{a}}. \tag{9}$$

This shows that the photon-added coherent states can be interpreted as nonlinear coherent states. These states are a class of nonlinear coherent states that can be physically realized in the interaction of a two-level atom with a cavity field that is initially prepared in the cohrent state $|\alpha\rangle$.

3. $|\alpha, m\rangle$ as deformed m-photon number state $|m\rangle$

In this section it is shown that the photon-added coherent states can be written as a nonunitarily deformed number state. This is achieved by the method given by Shanta *et al* [11]. First, a brief review of the method is given. Consider an 'annihilation operator' $\hat{A}$ which annihilates a set of number states $|n_i\rangle$, $i = 1, 2, \ldots, k$. Then we can construct a sector S_i by repeatedly applying $\hat{A}^\dagger$, the adjoint of $\hat{A}$, on the number state$|n_i\rangle$. Thus we have k sectors corresponding to the states that are annihilated by $\hat{A}$. A given sector may turn out to be either finite or infinite dimensional. If a sector, say S_j, is of infinite dimension then we construct an operator ${\hat{G}_j}^\dagger$ such that the commutator $[\hat{A}, {\hat{G}_j}^\dagger] = 1$ holds in that sector. Then the eigenstates of $\hat{A}$ can be written as $\mathrm{e}^{\alpha {\hat{G}_j}^\dagger}|n_j\rangle$. If the operator $\hat{A}$ is of the form $f(\hat{n})\hat{a}^p$, where p is a non-negative integer and $f(\hat{n})$ is a operator valued function of the number operator $\hat{a}^\dagger\hat{a}$, such that it annihilates the number state$|j\rangle$ then ${\hat{G}_j}^\dagger$ is constructed as

$$ {\hat{G}_j}^\dagger = \frac{1}{p}\hat{A}^\dagger \frac{1}{\hat{A}\hat{A}^\dagger}(\hat{a}^\dagger\hat{a} + p - j). \tag{10} $$

The photon-added coherent states are the eigenstates of $f(\hat{n}, m)\hat{a}$ with $f(\hat{n}, m)$ given by equation (9). The operator $f(\hat{n}, m)\hat{a}$ annihilates the vacuum state $|0\rangle$ and the m-photon state$|m\rangle$. The states in between the vacuum state and the m-photon state are not annihilated. In this sense it is different from the m-photon annihilation operator $\hat{a}^m$ which annihilates all the states$|i\rangle$, $i = 0, 1, 2, \ldots, m$. To discuss the case of $|\alpha, m\rangle$ let

$$ \hat{A} = \left(1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}}\right)\hat{a}. \tag{11} $$

The adjoint $\hat{A}^\dagger$ is given by

$$ \hat{A}^\dagger = \hat{a}^\dagger\left(1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}}\right). \tag{12} $$

We construct the sector S_0 by repeatedly applying $\hat{A}^\dagger$ on the vacuum state $|0\rangle$. S_0 is the set $|i\rangle, i = 0, 1, 2, \ldots, m-1$ and it is finite dimensional. The sector S_m, built by the repeated application of $\hat{A}^\dagger$ on $|m\rangle$, is the set $|i\rangle, i = m, m+1, \ldots$ and it is of infinite dimension. Hence we can construct an operator $\hat{G}^\dagger$ such that $[\hat{A}, \hat{G}^\dagger] = 1$ holds in S_m. To construct $\hat{G}^\dagger$, corresponding to the operator$\hat{A}$ given by equation (11), we set $p = 1$ and $j = m$ in equation (10) and this yields

$$ \hat{G}^\dagger = \hat{a}^\dagger. \tag{13} $$

Thus on the sector S_m we have $[\hat{A}, \hat{a}^\dagger] = 1$ and hence the photon-added coherent states, which are the eigenstates of $\hat{A}$, can be written as $\mathrm{e}^{\alpha\hat{a}^\dagger}|m\rangle$. However, this is not a unitary deformation.

4. Eigenstates of $f(\hat{n}, m)\hat{a}$ with negative m

The form of $f(\hat{n}, m)$, given by equation (9), suggests that it is a well-defined operator valued function, on the harmonic oscillator Hilbert space, also for negative integer values of m. In this section the nonlinear coherent states, with negative m in the expression for $f(\hat{n}, m)$, are constructed. Denoting the eigenstates by $|\alpha, -m\rangle$, the equation to determine them is

$$ \left(1 + \frac{m}{1 + \hat{a}^\dagger\hat{a}}\right)\hat{a}|\alpha, -m\rangle = \alpha|\alpha, -m\rangle. \tag{14} $$

Expanding $|\alpha, -m\rangle$ in terms of the number states $|n\rangle$ as

$$ |\alpha, -m\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{15} $$

where c_n are the expansion coefficients and substituting the expansion in equation (14) leads to the recursion relation

$$ c_n = \frac{m!\sqrt{n!}\alpha^n}{(n+m)!}c_0. \tag{16} $$

The constant c_0 is determined by normalization. The normalized $|\alpha, -m\rangle$ is given by

$$
\begin{aligned}
|\alpha, -m\rangle &= Nm! \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}(n+1)(n+2)\ldots(n+m)} |n\rangle; \\
N^{-1} &= m! \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2}} = \sqrt{{}_2F_2(1, 1, m+1, m+1, |\alpha|^2)}
\end{aligned} \tag{17}
$$

where ${}_2F_2(1, 1, m+1, m+1, |\alpha|^2)$ is the generalized hypergeometric function [12]. The number state expansion for the state $|\alpha, m\rangle$ is [10]

$$
|\alpha, m\rangle = \frac{\exp(-|\alpha|^2/2)}{\sqrt{L_m(-|\alpha|^2)m!}} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(m+n)!}}{n!} |n+m\rangle \tag{18}
$$

where $L_m(x)$ is the Lauguerre polynomial of order m defined by [12]

$$
L_m(x) = \sum_{n=0}^{m} \frac{(-x)^n m!}{(n!)^2 (m-n)!}. \tag{19}
$$

The state $|\alpha, -m\rangle$ involves a superpostion of all the Fock states starting with the vacuum state $|0\rangle$. But in the state $|\alpha, m\rangle$ the Fock states $|0\rangle, |1\rangle \ldots |m-1\rangle$ are not present. This important difference leads to different limiting cases of the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ when $\alpha \to 0$. In the limit $\alpha \to 0$ the state $|\alpha, -m\rangle$ becomes the vacuum state $|0\rangle$, which has properties like a classical radiation field, irrespective of the value of m and the state $|\alpha, m\rangle$ becomes the Fock state $|m\rangle$ which is nonclassical. In the limit $m \to 0$ the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become the coherent state $|\alpha\rangle$. Thus, $|\alpha, -m\rangle$ ($|\alpha, m\rangle$) is a state that is intermediate between the vacuum state (the number state $|m\rangle$) and the coherent state.

The photon-added coherent states are obtained by the action of $\hat{a}^{\dagger m}$ on the coherent state. The states $|\alpha, -m\rangle$ can be written in a similar form using the inverse operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger -1}$ [13]. These operators are defined in terms of their actions on the number states $|n\rangle$ as follows:

$$
\hat{a}^{-1}|n\rangle = \frac{1}{\sqrt{n+1}} |n+1\rangle \tag{20}
$$

$$
\hat{a}^{\dagger -1}|n\rangle = \frac{1}{\sqrt{n}} |n-1\rangle \qquad \text{for n} \neq 0 \tag{21}
$$

$$
\hat{a}^{\dagger -1}|0\rangle = 0. \tag{22}
$$

Using these inverse operators and equation (17) the state $|\alpha, -m\rangle$ can be written as

$$
|\alpha, -m\rangle = N\hat{a}^{\dagger -m}\hat{a}^{-m}|\alpha\rangle. \tag{23}
$$

The states $|\alpha, -m\rangle$ correspond to the nonlinear coherent states with $-m$ replacing m in $f(\hat{n}, m)$. However, they are obtained by the action of $\hat{a}^{\dagger -m}\hat{a}^{-m}$ on $|\alpha\rangle$ and not $\hat{a}^{\dagger -m}$ on $|\alpha\rangle$. Using the method reviewed in section 3 we can show that

$$
|\alpha, -m\rangle = \mathrm{e}^{\alpha \hat{G}^\dagger}|0\rangle \tag{24}
$$

where $\hat{G}^\dagger = \hat{a}^\dagger \frac{1+\hat{a}^\dagger\hat{a}}{1+m+\hat{a}^\dagger\hat{a}}$. The state $|\alpha, -m\rangle$ is obtained by deforming the vacuum state $|0\rangle$ while the state $|\alpha, m\rangle$ is obtained from the m-photon state $|m\rangle$.

5. Squeezing in $|\alpha, -m\rangle$

The state $|\alpha, -m\rangle$ exhibits squeezing in both x- and p-quadratures. The x- and p-quadratures are given interms of $\hat{a}$ and $\hat{a}^\dagger$ by

$$
x = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad p = \frac{\hat{a} - \hat{a}^\dagger}{\mathrm{i}\sqrt{2}}. \tag{25}
$$

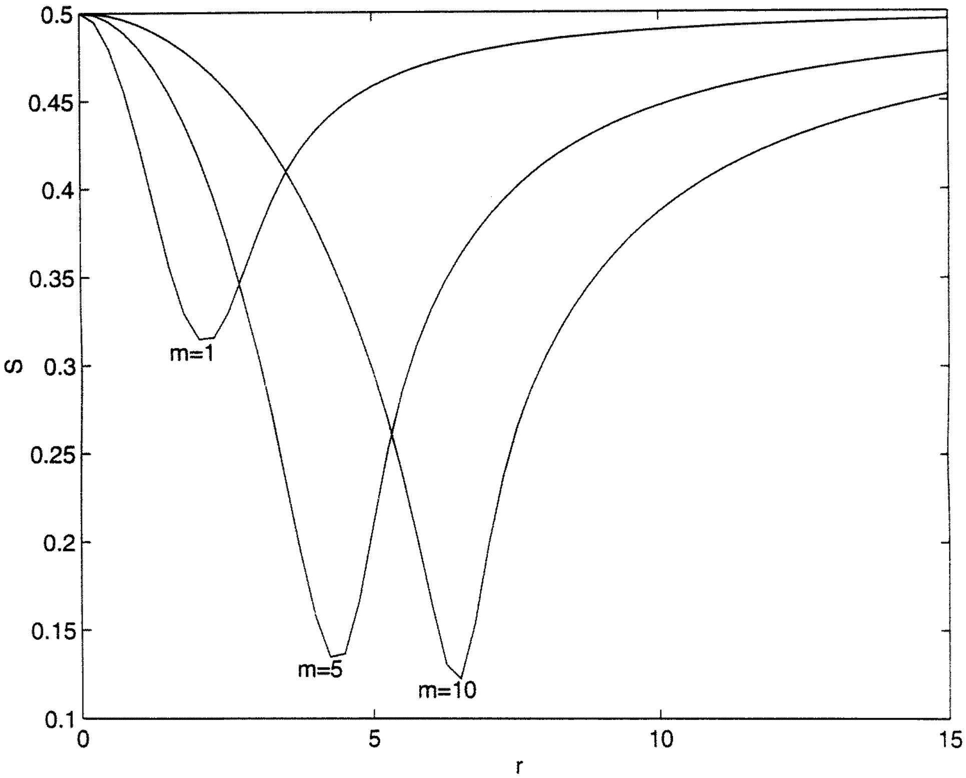

Figure 1. Uncertainty S, $\langle p^2\rangle - \langle p\rangle^2$, in p as a function of α for $m = 1$, $m = 5$ and $m = 10$ for the state $|\alpha, -m\rangle$. The real α is denoted as r.

The mean values of the relevant operators in the state $|\alpha, -m\rangle$ are

$$\langle \hat{a}\rangle = \alpha N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)}{(n+m+1)} \tag{26}$$

$$\langle \hat{a}^2\rangle = \alpha^2 N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)(n+2)}{(n+m+1)(n+m+2)} \tag{27}$$

and

$$\langle \hat{a}^\dagger \hat{a}\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n. \tag{28}$$

The mean values of $\hat{a}^\dagger$ and $\hat{a}^{\dagger 2}$ are obtained by taking the complex conjugates of $\langle \hat{a}\rangle$ and $\langle \hat{a}^2\rangle$ respectively. The uncertainty in x is

$$\langle x^2\rangle - \langle x\rangle^2 = \tfrac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}^2\rangle + \langle \hat{a}^{\dagger 2}\rangle - \langle \hat{a}\rangle^2 - \langle \hat{a}^\dagger\rangle^2 - 2\langle \hat{a}\rangle\langle \hat{a}^\dagger\rangle] \tag{29}$$

and that in p is

$$\langle p^2\rangle - \langle p\rangle^2 = \tfrac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^2\rangle - \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}\rangle^2 + \langle \hat{a}^\dagger\rangle^2 - 2\langle \hat{a}\rangle\langle \hat{a}^\dagger\rangle]. \tag{30}$$

In figure 1 the variance in p is shown for real α for various values of m. As expected the uncertainty in p is close to $\frac{1}{2}$, the uncertainty in p for the vacuum state, when α is close to zero. In the case of the state $|\alpha, m\rangle$ the variance is close to $m + \frac{1}{2}$ when α is close to zero. For

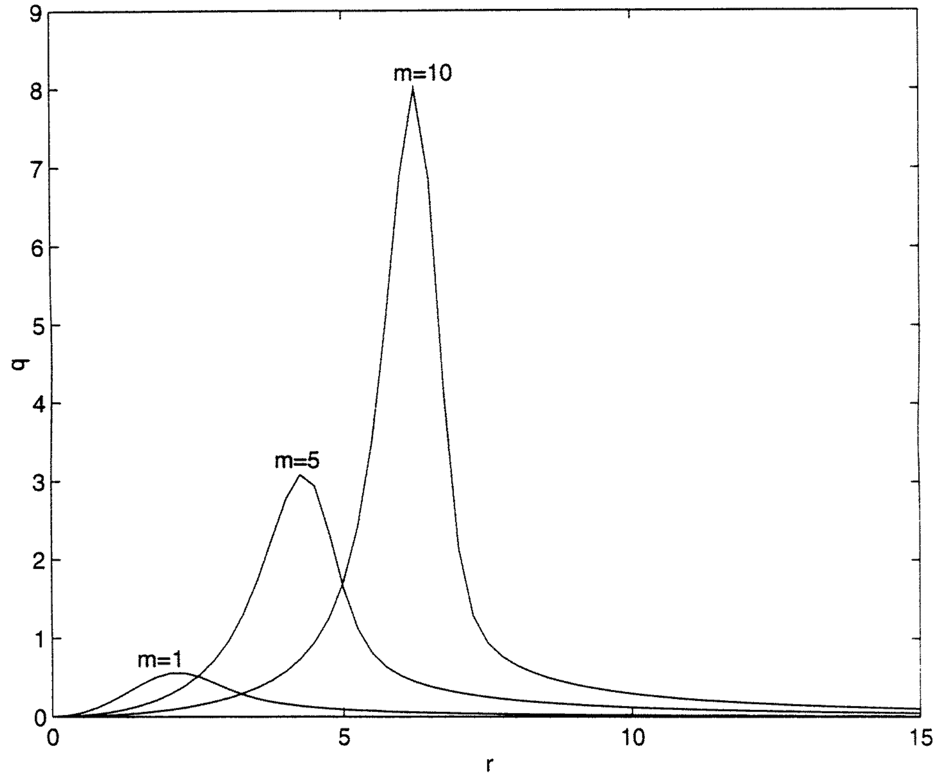

Figure 2. Mandel's q-parameter as a function of $|\alpha|$ for $m = 1$, $m = 5$ and $m = 10$. $|\alpha|$ is denoted as r.

real values of α the p-quadrature is always squeezed, $\langle p^2\rangle - \langle p\rangle^2 < \frac{1}{2}$, for the state $|\alpha, -m\rangle$. For large values of α the variance in p approaches that of the coherent state. If α becomes iα the expression for variance in p becomes the expression for variance in x. Since p shows squeezing for real α the x-quadrature exhibits squeezing for imaginary α.

6. Photon statistics of $|\alpha, -m\rangle$

The photon number distribution $p(n)$ for the state $|\alpha, -m\rangle$ is

$$\begin{aligned} p(n) &= |\langle n|\alpha, -m\rangle|^2 \\ &= N^2 m!^2 \frac{|\alpha|^{2n} n!}{(n+m)!^2}. \end{aligned} \tag{31}$$

When $m = 0$ the distribution becomes the Poissonian distribution whose mean value is $|\alpha|^2$.

A measure of the variance of the photon number distribution is given by Mandel's q-parameter [14],

$$q = \frac{\langle(\triangle\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}. \tag{32}$$

The coherent states have q as zero. A negative value of q indicates that the distribution $p(n)$ is sub-Poissonian and that it is a nonclassical feature. The photon-added coherent states $|\alpha, m\rangle$

are always sub-Poissonian for all values of m. For the state $|\alpha, -m\rangle$ the mean values of $\hat{n}$ and $\hat{n}^2$ are given by

$$\langle\hat{n}\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n \tag{33}$$

$$\langle\hat{n}^2\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n^2. \tag{34}$$

In figure 2 the q-parameter, calculated using equations (32)–(34), for the state $|\alpha, -m\rangle$ is shown as a function of $|\alpha|$. The states $|\alpha, -m\rangle$ have q always greater than zero indicating that they are super-Poissonian. For small values of α the q-parameter is close to zero and for large values of α it approaches zero.

7. Summary

The photon-added coherent states are nonlinear coherent states. They are the eigenstates of the operator $(1 - \frac{m}{1+\hat{a}^\dagger\hat{a}})\hat{a}$ when m takes positive integer values. This operator is also a meaningful operator when m takes negative integer values. The corresponding eigenstates $|\alpha, -m\rangle$ are nonclassical. The photon-added coherent state $|\alpha, m\rangle$ results from the action of $\hat{a}^{\dagger m}$ on the coherent state $|\alpha\rangle$ while the state $|\alpha, -m\rangle$ comes from the action of $\hat{a}^{\dagger -m}\hat{a}^{-m}$ on the coherent state $|\alpha\rangle$. Both $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ show squeezing. While $|\alpha, m\rangle$ is sub-Poissonian the state $|\alpha, -m\rangle$ is not sub-Poissonian. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become $|m\rangle$ and $|0\rangle$ respectively in the limit $\alpha \to 0$ but they become the coherent state $|\alpha\rangle$ when $m \to 0$. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ are the result of deforming the number states $|m\rangle$ and $|0\rangle$ respectively.

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